

# ZERO-ANNIHILATOR GRAPHS OF COMMUTATIVE RINGS

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**ABSTRACT.** Assume that  $R$  is a commutative ring with nonzero identity. In this paper, we introduce and investigate *zero-annihilator graph of  $R$*  denoted by  $\text{ZA}(R)$ . It is the graph whose vertex set is the set of all nonzero nonunit elements of  $R$  and two distinct vertices  $x$  and  $y$  are adjacent whenever

$$\text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}.$$

## 1. INTRODUCTION

Throughout this paper all rings are commutative with nonzero identity. In [6], Beck associated to a ring  $R$  its zero-divisor graph  $G(R)$  whose vertices are the zero-divisors of  $R$  (including 0), and two distinct vertices  $x$  and  $y$  are adjacent if  $xy = 0$ . Later, in [3], Anderson and Livingston studied the subgraph  $\Gamma(R)$  (of  $G(R)$ ) whose vertices are the nonzero zero-divisors of  $R$ . In the recent years, several researchers have done interesting and enormous works on this field of study. For instance, see [4, 5, 9]. The concept of co-annihilating ideal graph of a ring  $R$ , denoted by  $\mathcal{A}_R$  was introduced by Akbari et al. in [1]. As in [1], *co-annihilating ideal graph of  $R$* , denoted by  $\mathcal{A}_R$ , is a graph whose vertex set is the set of all non-zero proper ideals of  $R$  and two distinct vertices  $I$  and  $J$  are adjacent whenever  $\text{Ann}_R(I) \cap \text{Ann}_R(J) = \{0\}$ . In the present paper, we introduce *zero-annihilator graph of  $R$*  denoted by  $\text{ZA}(R)$ . It is the graph whose vertex set is the set of all nonzero nonunit elements of  $R$  and two distinct vertices  $x$  and  $y$  are adjacent whenever  $\text{Ann}_R(Rx + Ry) = \text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$ .

Let  $G$  be a simple graph with the vertex set  $V(G)$  and edge set  $E(G)$ . For every vertex  $v \in V(G)$ ,  $N_G(v)$  is the set  $\{u \in V(G) \mid uv \in E(G)\}$ . The *degree* of a vertex  $v$  is defined as  $\deg_G(v) = |N_G(v)|$ . The *minimum degree* of  $G$  is denoted by  $\delta(G)$ . Recall that a graph  $G$  is *connected* if there is a path between every two distinct vertices. For distinct vertices  $x$  and  $y$  of a connected graph  $G$ , let  $d_G(x, y)$  be the length of the shortest path from  $x$  to  $y$ . The *diameter* of a connected graph  $G$  is  $\text{diam}(G) = \sup\{d_G(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\}$ . The *girth* of  $G$ , denoted by  $\text{girth}(G)$ , is defined as the length of the shortest cycle in  $G$  and  $\text{girth}(G) = \infty$  if  $G$  contains no cycles. A *bipartite graph* is a graph all of whose vertices can be partitioned into two parts  $U$  and  $V$  such that every edge joins a vertex in  $U$  to a vertex in  $V$ . A *complete bipartite graph* is a bipartite graph with parts  $U, V$  such that every vertex in  $U$  is adjacent to every vertex in  $V$ . A graph in which all vertices have degree  $k$  is called a  *$k$ -regular graph*. A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. Also,

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if a graph  $G$  contains one vertex to which all other vertices are joined and  $G$  has no other edges, is called a *star graph*. A *clique* in a graph  $G$  is a subset of pairwise adjacent vertices and the number of vertices in a maximum clique of  $G$ , denoted by  $\omega(G)$ , is called the *clique number* of  $G$ . Obviously,  $\chi(G) \geq \omega(G)$ .

## 2. SOME PROPERTIES OF $\mathbf{ZA}(R)$

Recall that, an *empty graph* is a graph with no edges. A *Bézout ring* is a ring in which all finitely generated ideals are principal.

**Theorem 2.1.** *Let  $R$  be a ring. If  $\mathbf{ZA}(R)$  is an empty graph, then  $R$  is a local ring and  $\text{Ann}_R(x) \neq \{0\}$  for every nonunit element  $x \in R$ . The converse is true if  $R$  is a Bézout ring.*

*Proof.* Assume that  $\mathbf{ZA}(R)$  is empty. Let  $\mathfrak{m}_1, \mathfrak{m}_2$  be two distinct maximal ideals of  $R$ . Then  $\mathfrak{m}_1 + \mathfrak{m}_2 = R$  implies that there exist  $x \in \mathfrak{m}_1$  and  $x_2 \in \mathfrak{m}_2$  such that  $x + y = 1$ . So  $x$  and  $y$  are adjacent, which is a contradiction. Hence  $R$  is a local ring. Let  $\mathfrak{m}$  be the maximal ideal of  $R$  and  $x$  be an element of  $\mathfrak{m}$ . Suppose that  $\text{Ann}_R(x) = \{0\}$ . Then  $\{x^n | n \in \mathbb{N}\}$  is an infinite clique in  $\mathbf{ZA}(R)$  that is a contradiction. So  $\text{Ann}_R(x) \neq \{0\}$ .

Suppose that  $R$  is a local Bézout ring and  $\text{Ann}_R(x) \neq \{0\}$  for every nonunit element  $x \in R$ . Let  $x, y$  be two vertices in  $\mathbf{ZA}(R)$ . Then  $x, y \in \mathfrak{m}$ . Hence  $Rx + Ry = Rz$  for some nonzero nonunit element  $z \in R$ . So  $x, y$  are not adjacent which shows that  $\mathbf{ZA}(R)$  is empty.  $\square$

*Remark 2.2.* Suppose that  $R$  has a nontrivial idempotent element  $e$ . Then  $e + (1 - e) = 1$  implies that  $e$  and  $1 - e$  are adjacent. Hence  $\deg_{\mathbf{ZA}(R)}(e) \geq 1$  and so  $\mathbf{ZA}(R)$  is not an empty graph.

*Remark 2.3.* Let  $R$  be a ring. Notice that if  $R$  is an Artinian ring or a Boolean ring, then  $\dim(R) = 0$ . By [2, Theorem 3.4],  $\dim(R) = 0$  if and only if for every  $x \in R$  there exists a positive integer  $n$  such that  $x^{n+1}$  divides  $x^n$ . Therefore, every nonzero nonunit element of a zero-dimensional ring has a nonzero annihilator. Hence, if  $R$  is a zero-dimensional chained ring, then  $\mathbf{ZA}(R)$  is an empty graph.

Let  $Z^*(R)$  denote the zero divisors of  $R$  and  $Z(R) = Z^*(R) \cup \{0\}$ .

**Theorem 2.4.** *Let  $R$  be a ring and  $S$  be a multiplicative closed subset of  $R$  such that  $S \cap Z(R) = \{0\}$ . Then  $\mathbf{ZA}(R) \simeq \mathbf{ZA}(R_S)$ .*

*Proof.* Define the vertex map  $\Phi : V(\mathbf{ZA}(R)) \longrightarrow V(\mathbf{ZA}(R_S))$  by  $x \mapsto \frac{x}{1}$ . We can easily verify that  $x = y$  if and only if  $\frac{x}{1} = \frac{y}{1}$ . Also, it is easy to see that  $\text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$  if and only if  $\text{Ann}_{R_S}(\frac{x}{1}) \cap \text{Ann}_{R_S}(\frac{y}{1}) = \{\frac{0}{1}\}$ .  $\square$

**Theorem 2.5.** *Let  $R$  be a Bézout ring with  $|Max(R)| < \infty$  such that  $\delta(\mathbf{ZA}(R)) > 0$ . Then  $\mathbf{ZA}(R)$  is a finite graph if and only if every vertex of  $\mathbf{ZA}(R)$  has finite degree.*

*Proof.* The “only if” part is evident.

Suppose that each vertex of  $\mathbf{ZA}(R)$  has finite degree. If  $\text{Ann}_R(x) = \{0\}$  for some nonzero nonunit element  $x \in R$ , then  $x$  is adjacent to all vertices of  $\mathbf{ZA}(R)$  that

implies  $\mathbf{ZA}(R)$  is a finite graph. Assume that  $\text{Ann}_R(x) \neq \{0\}$  for each nonzero nonunit element  $x \in R$ . We claim that  $\text{Jac}(R) = \{0\}$ . On the contrary, assume that there exists a nonzero element  $a \in \text{Jac}(R)$ . Since  $\mathbf{ZA}(R)$  has no isolated vertex,  $a$  is adjacent to another vertex, say  $b$ . Since  $R$  is a Bézout ring,  $Ra + Rb$  is generated by a nonzero nonunit element  $c$  of  $R$  and so  $\text{Ann}_R(Ra + Rb) = \text{Ann}_R(c) \neq \{0\}$ , which is impossible. So  $\text{Jac}(R) = \{0\}$ . Hence by Chinese Remainder Theorem we have  $R \simeq F_1 \times F_2 \times \cdots \times F_n$ , where  $F_i$ 's are fields and  $n = |\text{Max}(R)|$ . Let  $0 \neq u \in F_1$ . Then  $(u, 0, \dots, 0)$  and  $(0, 1, \dots, 1)$  are adjacent. Since  $(0, 1, \dots, 1)$  has finite degree, so  $F_1$  is a finite field. Similarly we can show that  $F_i$ 's are finite fields. Consequently  $R$  has finitely many nonzero nonunit elements and the proof is complete.  $\square$

**Theorem 2.6.** *Let  $R$  be a Bézout ring with  $|\text{Max}(R)| < \infty$ . Then the following conditions are equivalent:*

- (1)  $\mathbf{ZA}(R)$  is a bipartite graph with  $\delta(\mathbf{ZA}(R)) > 0$ ;
- (2)  $\mathbf{ZA}(R)$  is a complete bipartite graph;
- (3)  $R \simeq F_1 \times F_2$  where  $F_1$  and  $F_2$  are two fields.

*Proof.* (1)  $\Rightarrow$  (3) Suppose that  $\mathbf{ZA}(R)$  is a bipartite graph with  $\delta(\mathbf{ZA}(R)) > 0$ . If  $\text{Ann}_R(x) = \{0\}$  for some nonzero nonunit element  $x$  of  $R$ , then  $\{x^n | n \in \mathbb{N}\}$  is an infinite clique that is a contradiction. Then, for every nonzero nonunit element  $x$  of  $R$  we have  $\text{Ann}_R(x) \neq \{0\}$ . Similar to the proof of Theorem 2.5 we can show that  $R = F_1 \times F_2 \times \cdots \times F_n$ , where  $F_i$ 's are fields and  $n = |\text{Max}(R)|$ . Clearly  $n \neq 1$ . If  $n \geq 3$ , then  $\{(0, 1, \dots, 1), (1, 0, 1, \dots, 1), (1, 1, 0, 1, \dots, 1)\}$  is a clique in  $\mathbf{ZA}(R)$ , a contradiction. So  $R \simeq F_1 \times F_2$ .

(3)  $\Rightarrow$  (2) Suppose that  $R \simeq F_1 \times F_2$  where  $F_1$  and  $F_2$  are two fields. Every vertex in  $\mathbf{ZA}(R)$  is of the form  $(u, 0)$  or  $(0, v)$  where  $0 \neq u \in F_1$  and  $0 \neq v \in F_2$ . Also, two vertices  $(u, 0)$  and  $(0, v)$  are adjacent. On the other hand, every two vertices  $(u_1, 0), (u_2, 0)$  cannot be adjacent.

(2)  $\Rightarrow$  (1) is clear.  $\square$

**Theorem 2.7.** *Let  $R$  be a ring and  $n \geq 2$  be a natural number. Then*

$$\text{girth}(\mathbf{ZA}(M_n(R))) = 3.$$

*Proof.* For  $n = 2$ , the following matrices are pairwise adjacent in  $\mathbf{ZA}(M_2(R))$ :

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

For  $n \geq 3$ , the following matrices are pairwise adjacent in  $\mathbf{ZA}(M_n(R))$ :

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

□

### 3. WHEN IS $\mathbf{ZA}(R)$ CONNECTED?

A ring  $R$  is called *semiprimitive* if  $\text{Jac}(R) = 0$ , [7]. A ring  $R$  is semiprimitive if and only if it is a subdirect product of fields, [8, p. 179].

**Theorem 3.1.** *Let  $R$  be a semiprimitive ring. If at least one of the maximal ideals of  $R$  is principal, then  $\mathbf{ZA}(R)$  is a connected graph with  $\text{diam}(\mathbf{ZA}(R)) \leq 4$ .*

*Proof.* Suppose that  $\mathfrak{m}$  is a maximal ideal of  $R$  where  $\mathfrak{m} = Rt$  for some  $t \in R$ . Let  $x, y$  be two different nonzero nonunit elements of  $R$ . Consider the following cases:

**Case 1.** Let  $x, y \notin \mathfrak{m}$ . Then  $Rx + \mathfrak{m} = R$  and  $Ry + \mathfrak{m} = R$ . Hence  $x, y$  are adjacent to  $t$ . So  $d_{\mathbf{ZA}(R)}(x, y) \leq 2$ .

**Case 2.** Let  $x \in \mathfrak{m}$  and  $y \notin \mathfrak{m}$ . Notice that  $y$  is adjacent to  $t$ . Since  $\text{Jac}(R) = \{0\}$ , there exists a maximal ideal  $\mathfrak{m}'$  different from  $\mathfrak{m}$  such that  $x \notin \mathfrak{m}'$ . So  $Rx + \mathfrak{m}' = R$ , and thus there exist elements  $r \in R$  and  $z \in \mathfrak{m}'$  such that  $rx + z = 1$ . Therefore  $\text{Ann}_R(x) \cap \text{Ann}_R(z) = \{0\}$ . So  $x$  is adjacent to  $z$ . Clearly  $z \notin \mathfrak{m}$ . Then  $z$  is adjacent to  $t$ . Hence  $d_{\mathbf{ZA}(R)}(x, y) \leq 3$ .

**Case 3.** Let  $x, y \in \mathfrak{m}$ . A manner similar to Case 2 shows that  $d_{\mathbf{ZA}(R)}(x, t) \leq 2$  and  $d_{\mathbf{ZA}(R)}(y, t) \leq 2$ . Therefore  $d_{\mathbf{ZA}(R)}(x, y) \leq 4$ .

Consequently  $\mathbf{ZA}(R)$  is a connected graph with  $\text{diam}(\mathbf{ZA}(R)) \leq 4$ . □

**Theorem 3.2.** *Let  $R$  be a Bézout ring. If  $\mathbf{ZA}(R)$  is connected, then one of the following conditions holds:*

- (1) *There exists a nonzero nonunit element  $x$  of  $R$  such that  $\text{Ann}_R(x) = \{0\}$ ,*
- (2)  *$\text{Jac}(R) = \{0\}$ ,*
- (3)  *$\text{Jac}(R) = \{0, x\}$  where  $x$  is the only nonzero nonunit element of  $R$ .*

*Proof.* Assume that for every nonzero nonunit element  $x$  of  $R$ ,  $\text{Ann}_R(x) \neq \{0\}$  and also  $\text{Jac}(R) \neq \{0\}$ . Let  $x$  be a nonzero element in  $\text{Jac}(R)$ . Suppose that  $\mathbf{ZA}(R)$  has a vertex  $y$  different from  $x$ . Thus  $Rx + Ry = Rz$  for some  $z \in R$ , because  $R$  is a Bézout ring. Notice that  $y \in \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$  of  $R$ . Hence  $z$  is nonzero nonunit and so by assumption  $\text{Ann}_R(z) \neq \{0\}$ , which shows that  $x$  and  $y$  are not adjacent. This contradiction implies that  $|V(\mathbf{ZA}(R))| = 1$ , and so  $\text{Jac}(R) = \{0, x\}$ . □

As a direct consequence of Theorem 3.1 and Theorem 3.2 we have the following result.

**Corollary 3.3.** *Let  $R$  be a Bézout ring such that at least one of the maximal ideals of  $R$  is principal. Then  $\mathbf{ZA}(R)$  is connected if and only if one of the following conditions holds:*

- (1) *There exists a nonzero nonunit element  $x$  of  $R$  such that  $\text{Ann}_R(x) = \{0\}$ ,*
- (2)  *$\text{Jac}(R) = \{0\}$ ,*
- (3)  *$\text{Jac}(R) = \{0, x\}$  where  $x$  is the only nonzero nonunit element of  $R$ .*

**Theorem 3.4.** *Let  $R = F_1 \times F_2 \times \cdots \times F_n$  where  $F_i$ 's are fields. Then  $\text{ZA}(R)$  is a connected graph with*

$$\text{diam}(\text{ZA}(R)) = \begin{cases} 1 & \text{if } n = 2 \text{ and } |F_1| = |F_2| = 2 \\ 2 & \text{if } n = 2 \text{ and either } |F_1| > 2 \text{ or } |F_2| > 2 \\ 3 & \text{if } n \geq 3. \end{cases}$$

*Proof.* Let  $n = 2$ . In this case every vertex in  $\text{ZA}(R)$  is of the form  $(u, 0)$  or  $(0, v)$  where  $u \neq 0$  and  $v \neq 0$ . Furthermore, two vertices  $(u, 0)$  and  $(0, v)$  are adjacent. In the case when  $n = 2$  and  $|F_1| = |F_2| = 2$ , we have  $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ . So  $\text{ZA}(R) \simeq K_2$ . Let  $n = 2$  and  $|F_1| > 2$ . In this case, every two different vertices  $(u_1, 0)$  and  $(u_2, 0)$  cannot be adjacent. On the other hand  $(u_1, 0)$  and  $(u_2, 0)$  are adjacent to  $(0, 1)$ . So  $d_{\text{ZA}(R)}((u_1, 0), (u_2, 0)) = 2$ . Hence  $\text{diam}(\text{ZA}(R)) = 2$ .

Now, let  $n \geq 3$ . Assume that  $u = (u_1, u_2, \dots, u_n)$  and  $v = (v_1, v_2, \dots, v_n)$  are two different vertices. There exist two indexes  $i, j$  such that  $u_i \neq 0$  and  $v_j \neq 0$ . So

$u = (u_1, u_2, \dots, u_n)$  is adjacent to  $(1, \dots, 1, \overbrace{0}^{i\text{-th}}, 1, \dots, 1)$ . Also  $v = (v_1, v_2, \dots, v_n)$  is adjacent to  $(1, \dots, 1, \overbrace{0}^{j\text{-th}}, 1, \dots, 1)$ . Furthermore, if  $i \neq j$ , then the vertex  $(1, \dots, 1, \overbrace{0}^{i\text{-th}}, 1, \dots, 1)$  is adjacent to  $(1, \dots, 1, \overbrace{0}^{j\text{-th}}, 1, \dots, 1)$ . Thus  $\text{ZA}(R)$  is connected and  $d_{\text{ZA}(R)}(u, v) \leq 3$ . In special case, we have the following path

$$(0, 1, 0, \dots, 0) - (1, 0, 1, \dots, 1) - (0, 1, \dots, 1) - (1, 0, \dots, 0).$$

Consequently  $\text{diam}(\text{ZA}(R)) = 3$ . □

#### 4. WHEN IS $\text{ZA}(R)$ STAR?

**Lemma 4.1.** *Let  $R$  be a ring. If  $\text{ZA}(R)$  is a star, then  $|\text{Max}(R)| \leq 2$ .*

*Proof.* Suppose that  $\text{ZA}(R)$  is a star. If  $\mathfrak{m}$  and  $\mathfrak{m}'$  is two different maximal ideals of  $R$ , then for every  $x \in \mathfrak{m} \setminus \mathfrak{m}'$  we have  $Rx + \mathfrak{m}' = R$ . Hence there exist elements  $r \in R$  and  $y \in \mathfrak{m}' \setminus \mathfrak{m}$  such that  $rs + y = 1$ . Therefore  $\text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$ . So  $x$  and  $y$  are adjacent. Let  $\mathfrak{m}_1, \mathfrak{m}_2$  and  $\mathfrak{m}_3$  be three different maximal ideals of  $R$ . Then there are elements  $a \in \mathfrak{m}_1 \setminus (\mathfrak{m}_2 \cup \mathfrak{m}_3)$ ,  $b \in \mathfrak{m}_2 \setminus (\mathfrak{m}_1 \cup \mathfrak{m}_3)$  and  $c \in \mathfrak{m}_3 \setminus (\mathfrak{m}_1 \cup \mathfrak{m}_2)$ . Then either  $a, b, c$  are pairwise adjacent or there exist at least two disjoint edges in  $\text{ZA}(R)$ , which is a contradiction. Consequently  $|\text{Max}(R)| \leq 2$ . □

**Theorem 4.2.** *Let  $R$  be a Bézout ring that is not a field. Then  $\text{ZA}(R)$  is a star if and only if one of the following conditions holds:*

- (1)  *$(R, \mathfrak{m})$  when  $\mathfrak{m} = \{0, x\}$  in which  $x$  is a nonzero element of  $R$  with  $x^2 = 0$ ,*
- (2)  *$R \simeq \mathbb{Z}_2 \times F$  where  $F$  is a field.*

*Proof.* ( $\Rightarrow$ ) Suppose that  $\text{ZA}(R)$  is a star. Hence  $|\text{Max}(R)| \leq 2$ , by Lemma 4.1. Notice that if  $\text{Ann}_R(t) = \{0\}$  for some element  $t$  of a maximal ideal  $\mathfrak{m}$ , then  $\{t^n | n \in \mathbb{N}\}$  is an infinite clique that is impossible. Consider the following cases:

**Case 1.**  $\text{Max}(R) = \{\mathfrak{m}\}$ . Let  $x$  be a nonzero element in  $\mathfrak{m}$ . Then by Theorem 2.1,  $\text{ZA}(R)$  is empty and so  $\mathfrak{m} = \{0, x\}$ . On the other hand, by Nakayama's Lemma we have that  $x^2 = 0$ .

**Case 2.**  $\text{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ . Since  $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ , there exist  $x \in \mathfrak{m}_1$  and  $y \in \mathfrak{m}_2$  such that  $x + y = 1$ . Hence  $x$  and  $y$  are adjacent. Now, if there exists  $0 \neq z \in \mathfrak{m}_1 \cap \mathfrak{m}_2$ , then  $z$  is not adjacent to  $x$  and  $y$ , because  $R$  is a Bézout ring and  $\text{Ann}_R(t) = \{0\}$  for every nonzero nonunit element  $t$  of  $R$ . This contradiction shows that  $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \{0\}$ . Hence by Chinese Remainder Theorem we deduce that  $R \simeq R/\mathfrak{m}_1 \oplus R/\mathfrak{m}_2$ . If there exist nonzero elements  $a_1, a_2 \in R/\mathfrak{m}_1$  and  $b_1, b_2 \in R/\mathfrak{m}_2$ , then we have the following path

$$(a_1, 0) - (0, b_1) - (a_2, 0) - (0, b_2),$$

a contradiction. Hence we can assume that  $R/\mathfrak{m}_1 = \mathbb{Z}_2$ .

( $\Leftarrow$ ) If (1) holds, the clearly  $\text{ZA}(R)$  is a star. Assume that (2) holds. Notice that  $(1, 0)$  is adjacent to all vertices  $(0, u)$  where  $u$  is a nonzero element of  $F$ . Also, for every two different elements  $u_1, u_2 \in F$ ,  $(0, u_1)$  and  $(0, u_2)$  are not adjacent. Consequently  $\text{ZA}(R)$  is a star.  $\square$

## 5. WHEN IS $\text{ZA}(R)$ COMPLETE?

**Proposition 5.1.** *Let  $R$  be a ring. If  $\text{ZA}(R)$  is a complete graph, then  $\mathcal{A}_R$  is a complete graph.*

*Proof.* Assume that  $\text{ZA}(R)$  is a complete graph. Let  $I, J$  be two nonzero proper ideals of  $R$ . Then there are two different nonzero nonunit elements  $x, y \in R$  such that  $x \in I$  and  $y \in J$ . Hence  $\text{Ann}_R(I) \cap \text{Ann}_R(J) \subseteq \text{Ann}_R(x) \cap \text{Ann}_R(y) = \{0\}$ . Therefore  $I$  and  $J$  are adjacent.  $\square$

The following remark shows that the converse of Proposition 5.1 is not true.

*Remark 5.2.* Consider the ring  $R = \mathbb{Z}_5 \times \mathbb{Z}_5$ . By [1, Theorem 6],  $\mathcal{A}_R (= K_2)$  is a complete graph. But  $\text{ZA}(R)$  is a 4-regular graph that is not a complete graph.

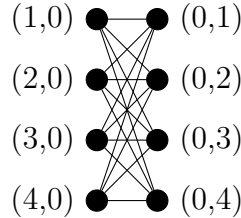


Fig. 1.  $\text{ZA}(R)$

**Theorem 5.3.** *Let  $R$  be a ring. Then  $\text{ZA}(R)$  is a complete graph if and only if one of the following conditions holds:*

- (1)  $R$  has exactly one nonzero nonunit element,
- (2)  $R$  is an integral domain,
- (3)  $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ .

*Proof.* ( $\Rightarrow$ ) Assume that  $\mathbf{ZA}(R)$  is a complete graph. Then, by Proposition 5.1,  $\mathcal{A}_R$  is a complete graph. Suppose that  $R$  is not an integral domain. So there exists a nonzero nonunit element  $x \in R$  such that  $\text{Ann}_R(x) \neq \{0\}$ . Therefore, [1, Theorem 6] implies that either  $R$  has exactly one nonzero proper ideal or  $R$  is a direct product of two fields. Suppose that the former case holds. If  $y$  is a nonzero nonunit element of  $R$  different from  $x$ , then  $Rx = Ry$ . So  $\text{Ann}_R(x) \cap \text{Ann}_R(y) = \text{Ann}_R(x) \neq \{0\}$ , which is a contradiction. Therefore  $R$  has exactly one nonzero nonunit element. Now, let  $R$  be a direct product of two fields, say  $R = F_1 \times F_2$ . If there exist two different nonzero elements  $u, v$  in  $F_1$ , then  $(u, 0)$  and  $(v, 0)$  cannot be adjacent. Hence  $F_1 = \mathbb{Z}_2$ . Similarly, we can show that  $F_2 = \mathbb{Z}_2$ . Consequently  $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ . ( $\Leftarrow$ ) Clearly, if (1) or (2) holds, then  $\mathbf{ZA}(R)$  is a complete graph. Assume that (3) holds. Then  $\mathbf{ZA}(R) \simeq K_2$  and we are done.  $\square$

## 6. WHEN IS $\mathbf{ZA}(R)$ $k$ -REGULAR?

Recall that a finite field of order  $q$  exists if and only if the order  $q$  is a prime power  $p^s$ . A finite field of order  $p^s$  is denoted by  $\mathbb{F}_{p^s}$ .

**Theorem 6.1.** *Let  $R$  be a Bézout ring with  $|\text{Max}(R)| < \infty$ . Then  $\mathbf{ZA}(R)$  is a  $k$ -regular graph ( $0 < k < \infty$ ) if and only if  $R \simeq \mathbb{F}_{k+1} \times \mathbb{F}_{k+1}$ .*

*Proof.* The “if” part has a routine verification. Let  $\mathbf{ZA}(R)$  be a  $k$ -regular graph ( $0 < k < \infty$ ). If  $\text{Ann}_R(x) = \{0\}$  for some nonzero nonunit element  $x$  of  $R$ , then  $\{x^n | n \in \mathbb{N}\}$  is an infinite clique that is a contradiction. Then, for every nonzero nonunit element  $x$  of  $R$  we have  $\text{Ann}_R(x) \neq \{0\}$ . Similar to the manner that described in the proof of Theorem 2.5, we have  $R \simeq F_1 \times F_2 \times \cdots \times F_n$  where  $F_i$ 's are fields and  $n = |\text{Max}(R)|$ . Since  $\text{Ann}_R((1, 0, \dots, 0)) = 0 \times F_2 \times F_3 \times \cdots \times F_n$  and  $\text{Ann}_R((0, 1, 0, \dots, 0)) = F_1 \times 0 \times F_3 \times \cdots \times F_n$ , then

$$N_{\mathbf{ZA}(R)}((1, 0, \dots, 0)) = \{(0, u_2, \dots, u_n) | u_i \in F_i \setminus \{0\} \text{ for } 2 \leq i \leq n\}$$

and

$$N_{\mathbf{ZA}(R)}((0, 1, 0, \dots, 0)) = \{(u_1, 0, u_3, \dots, u_n) | u_i \in F_i \setminus \{0\} \text{ for } 1 \leq i \leq n, i \neq 2\}.$$

So

$$(|F_2| - 1)(|F_3| - 1) \cdots (|F_n| - 1) = (|F_1| - 1)(|F_3| - 1) \cdots (|F_n| - 1),$$

because  $\mathbf{ZA}(R)$  is  $k$ -regular. Hence  $|F_1| = |F_2|$ . Similarly we can show that  $|F_1| = |F_2| = \cdots = |F_n|$ . Let  $n \geq 3$ . Note that  $N_{\mathbf{ZA}(R)}((1, 1, 0, \dots, 0))$  is the union of the following sets

$$\begin{aligned} &\{(u_1, 0, u_3, \dots, u_n) | u_i \in F_i \setminus \{0\} \text{ for } 1 \leq i \leq n, i \neq 2\}, \\ &\{(0, u_2, \dots, u_n) | u_i \in F_i \setminus \{0\} \text{ for } 2 \leq i \leq n\} \end{aligned}$$

and

$$\{(0, 0, u_3, \dots, u_n) | u_i \in F_i \setminus \{0\} \text{ for } 3 \leq i \leq n\}.$$

Therefore

$$(|F_1| - 1)^{n-1} = 2(|F_1| - 1)^{n-1} + (|F_1| - 1)^{n-2},$$

since  $\mathbf{ZA}(R)$  is  $k$ -regular. Thus  $|F_1| = 0$  which is a contradiction. Consequently  $n = 2$ . If there exist two different nonzero elements  $u, u'$  in  $F_1$ , then  $(u, 0)$  and  $(u', 0)$  cannot be adjacent. On the other hand for every nonzero elements  $u \in F_1$



and  $v \in F_2$ ,  $(u, 0)$  and  $(0, v)$  are adjacent. So  $\deg_{\mathbf{ZA}(R)}((u, 0)) = |F_1| - 1 = k$ . Therefore  $R \simeq \mathbb{F}_{k+1} \times \mathbb{F}_{k+1}$ .  $\square$

**Corollary 6.2.** *Let  $R$  be a Bézout ring with  $|\text{Max}(R)| < \infty$ . If  $\mathbf{ZA}(R)$  is a  $k$ -regular graph ( $0 < k < \infty$ ), then  $k + 1$  is a prime power.*

## 7. CHROMATIC NUMBER AND CLIQUE NUMBER OF $\mathbf{ZA}(R)$

Recall that, a ring  $R$  is said to be *reduced* if it has no nonzero nilpotent elements.

**Theorem 7.1.** *If  $R$  is a reduced Noetherian ring, then the chromatic number of  $\mathbf{ZA}(R)$  is infinite or  $R$  is a direct product of finitely many fields.*

*Proof.* The proof is similar to that of [1, Theorem 16].  $\square$

**Lemma 7.2.** *Let  $P_1$  and  $P_2$  be two prime ideals of a ring  $R$  with  $P_1 \cap P_2 = \{0\}$ . Then every two nonzero elements  $x \in P_1$  and  $y \in P_2$  are adjacent.*

*Proof.* Suppose that  $r \in \text{Ann}_R(x) \cap \text{Ann}_R(y)$ . Since  $rx = 0 \in P_2$  and  $x \notin P_2$ , then  $r \in P_2$ . Similarly it turns out that  $r \in P_1$ . Hence  $r \in P_1 \cap P_2 = \{0\}$ .  $\square$

**Theorem 7.3.** *Let  $R$  be a ring and  $n \geq 2$  be a natural number. If either  $|\text{Min}(R)| = n$  or  $R = R_1 \times R_2 \times \cdots \times R_n$  where  $R_i$ 's are rings, then  $\omega(\mathbf{ZA}(R)) \geq n$ .*

*Proof.* Assume that  $\text{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$  where  $\mathfrak{p}_i$ 's are nonzero. So, by Lemma 7.2,  $n \leq \omega(\mathbf{ZA}(R))$ . Now, suppose that  $R = R_1 \times R_2 \times \cdots \times R_n$  where  $R_i$ 's are rings. Then  $\{(1, \dots, 1, \overbrace{0}^{i\text{-th}}, 1, \dots, 1) | 1 \leq i \leq n\}$  is a clique in  $\mathbf{ZA}(R)$  and the result follows.  $\square$

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